B.sc math(H) part3 paper6

Toplc:Direct product of two group

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External direct products

Definition : If G_1 and G_2 be two groups then the set of all ordered pairs $\{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$ is called the external direct product of G_1 by G_2. The external direct product of G_1 by G_2 is written as $G_1 \times G_2$.

$\because \quad G_1 G_2 = \{g_1 g_2 : g_1 \in G_2, g_2 \in G\}$,

it should be obviously understood that

$$G_1 \times G_2 \neq G_2 \times G_1 \quad \text{and} \quad G_1 \times G_2 \neq G_1 G_2.$$

***Theorem 1.** If G_1 and G_2 be any two abstract groups then the set $G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2]$ is a group with respect to the binary operation denoted multiplicatively and defined as*

$$(g_1, g_2)(h_1, h_2) = (g_1 h_1, g_2 h_2)$$

where $g_1 h_1 \in G_1$, $g_2 h_2 \in G_2$.

Proof : To prove $G_1 \times G_2$ is a group, we have to satisfy the following properties :

(i) Closure Property : Since G_1 and G_2 be the two groups.

$\therefore \quad g_1, h_1 \in G_1 \Rightarrow g_1 h_1 \in G_1$,

and $g_2, h_2 \in G_2 \Rightarrow g_2 h_2 \in G_2$.

$\therefore \quad (g_1 h_1, g_2 h_2) \in G_1 \times G_2$

$\therefore \quad G_1 \times G_2$ is closed.

(ii) Associativity :

$$\begin{aligned}[(g_1, g_2)(h_1, h_2)](k_1, k_2) &= (g_1 h_1, g_2 h_2)(k_1, k_2) \\ &= [(g_1 h_1) k_1, (g_2 h_2) k_2] \\ &= [g_1 (h_1 k_1), g_2 (h_2 k_2)] \\ &= (g_1, g_2)(h_1 k_1, h_2 k_2) \\ &= (g_1, g_2)[(h_1, h_2)(k_1, k_2)].\end{aligned}$$

Hence the composition is associative.

(iii) Existence of identity : If e_1, e_2 be the identities of groups G_1 and G_2 respectively, then

$$(e_1, e_2) \in G_1 \times G_2.$$

Also $g_1 e_1 = e_1 g_1 = g_1$

and $g_2 e_2 = e_2 g_2 = g_2$.

Now $(g_1, g_2)(e_1, e_2) = (g_1 e_1, g_2 e_2) = (g_1, g_2)$

and $(e_1, e_2)(g_1, g_2) = (e_1 g_1, e_2 g_2) = (g_1, g_2)$

$\therefore$ $(e_1, e_2) \in G_1 \times G_2$ is the identity element of $G_1 \times G_2$.

(iv) Existence of inverse : Let $g_1, g_2 \in G_1$. Then

$$g_1 \in G_1 \Rightarrow g_1^{-1} \in G_1$$

$$g_2 \in G_2 \Rightarrow g_2^{-1} \in G_2$$

and $(g_1^{-1}, g_2^{-1}) \in G_1 \times G_2$

Also $g_1 g_1^{-1} = g_1^{-1} g_1 = e_1$

and $g_2 g_2^{-1} = g_2^{-1} g_2 = e_2$

$\because$ $(g_1, g_2)(g_1^{-1}, g_2^{-1}) = (g_1^{-1}, g_2^{-1})(g_1, g_2)$

$= (e_1, e_2)$ the identity, by (1).

$\Rightarrow$ $(g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-1}) \in G_1 \times G_2$.

Thus the inverse exists and belongs to the set.

Hence $G_1 \times G_2$ is a group with respect to the binary composition.

***Theorem 2.** If G_1 and G_2 are groups then the sub-set $G_1 \times \{e_2\}$ and $\{e_1\} \times G_2$ of $G_1 \times G_2$ are normal sub-groups of $G_1 \times G_2$ isomorphic to G_1 and G_2 respectively.*

Proof : $G_1 \times \{e_2\}$ is a sub-group of $G_1 \times G_2$.

If H be a sub-group, then $a \in H$, $b \in H \Rightarrow ab^{-1} \in H$.

Choose $a = (g_1, e_2)$, $b = (h_1, e_2) \in G_1 \times \{e_2\}$, then

$$ab^{-1} = (g_1, e_2)(h_1, e_2)^{-1} = (g_1, e_2)(h_1^{-1}, e_2^{-1})$$

$$= (g_1, e_2)(h_1^{-1}, e_2) = (g_1 h_1^{-1}, e_2 e_2)$$

$$= (g_1 h_1^{-1}, e_2) \in G_1 \times \{e\}.$$

$\because$ $g_1 \in G_1$, $h_1 \in G_1 \Rightarrow h_1^{-1} \in G_1$,

hence $g_1 h_1^{-1} \in G_1$.

This relation shows that $G_1 \times \{e_2\}$ is a sub-group of $G_1 \times G_2$.

Similarly we can prove that $\{e_1\} \times G_2$ is a sub-group of $G_1 \times G_2$. Now we are to prove that $G_1 \times \{e_2\}$ is a normal sub-group of $G_1 \times G_2$.

A sub-group H of G is a normal sub-group of G if for $h \in H$ $x h x^{-1} \in H$, $\forall x \in G$.

Here $H = G_1 \times \{e_2\}$ so that $h \in H$ is $\{g_1, e_2\}$

$G = G_1 \times G_2$ so that $x \in G_1 \times G_2$ is (p, q), $p \in G_1$, $q \in G_2$.

Now $xhx^{-1} = (p, q)(g_1, e_2)(p^{-1}, q^{-1})$

$= (pg\,p^{-1}, qe_2\,q^{-1})$

$= (p\,g\,p^{-1}, e_2) \in G_1 \times \{e_2\}$

because $p, g_1, p^{-1} \in G_1 \Rightarrow pg_1p^{-1} \in G_1$ and $qe_2q^{-1} = e_2$.

Hence $G_1 \times \{e_2\}$ is a normal sub-group of $G_1 \times G_2$.

Similarly we can prove that $\{e_1\} \times G_2$ is a normal sub-group of $\times G_2$.

Now we are to prove that $G_1 \times \{e_2\}$ is isomorphic to G_1

Let $f: G_1 \times \{e_2\} \rightarrow G_1$ defined as

$$f(g_1, e_2) = g_1.$$

Obviously f is one-one as

$$f(g_1, e_2) = f(h_1, e_2) \Rightarrow g_1 = h_1$$

$$\Rightarrow (g_1, e_2) = (h_1, e_2).$$

Also f is onto.

Again $f[(g_1, e_2)(h_1, e_2)] = f(g_1 h_1, e_2) = g_1 h_1$

$= f(g_1, e_2) f(h_1, e_2)$

i.e., $f(a, b) = f(a) f(b) \forall\ a, b \in G_1 \times \{e_2\}$.

Hence $G_1 \times \{e_2\} \equiv G_1$.

Similarly, we can prove that $\{e_2\} \times G_2 \equiv G_2$.

Theorem 3. *If G_1 and G_2 are groups, then*

(i) $G_1 \times \{e_1\} \cap \{e_2\} \times G_2 = (e_1, e_2)$,

i.e. the identity of $G_1 \times G_2$.

(ii) Every element of $G_1 \times \{e_2\}$ commutes with every lement of $\{e_1\} \times G_2$.

(iii) Every element of $G_1 \times G_2$ can be uniquely expressed is the product of an element of $G_1 \times \{e_2\}$ by an element of $\{e_1\} \times G_2$.

Proof : (i) Let $x \in A \cap B \Rightarrow x \in A$ and $x \in B$.

$\therefore\ x \in G_1 \times \{e_2\} \Rightarrow x \in (g_1, e_2)$.

This x can belong to $\{e_1\} \times e_2$ if $g_1 = e_1$, because in this case

$$x = (e_1, e_2) \in \{e_1\} \times G_2.$$

Thus (e_1, e_2) which is identity of group $G_1 \times G_2$ is the only element common to $G_1 \times \{e_2\}$ and $\{e_1\} \times G_2$;

i.e., $G_1 \times \{e_1\} \cap \{e_2\} \times G_2 = (e_1, e_2)$.

(ii) Let us consider $a \in G_1 \times \{e_2\}$, $\therefore$ $a = (g_1, e_2)$, $g_1 \in G_1$

and $b \in \{e_1\} \times G_2$, $\therefore$ $b = (e_1, g_2)$; $g_2 \in G_2$.

$\therefore$ $ab = (g_1, e_2)(g_1 e_1, e_2 g_2) = (g_1, g_2)$

and $ba = (e_1, g_2)(g_1, e_2) = (e_1 g_1, g_2 e_2) = (g_1, g_2)$.

$\therefore$ $ab = ba$.

Thus every element of $G_1 \times \{e_2\}$ commutes with every element of $\{e_1\} \times G_2$.

(iii) Suppose (g_1, g_2) be any element of $G_1 \times G_2$, then

$$(g_1, g_2) = (g_1 e_1, e_2 g_2) = (g_1, e_2)(e_1, g_2)$$

= product of an element of $G_1 \times \{e_2\}$ by an element of $\{e_1\} \times G_2$

This relation shows that there is at least one representation.

If $(g_1, g_2) = (h_1, e_2)(e_1, h_2)$ be another representation,

then $(g_1, g_2) = (h_1 e_1, e_2 h_2) = (h_1, h_2)$

$\Rightarrow g_1 = h_1$ and $g_2 = h_2$.

Hence the representation is unique.